Globalization for partial H-bicomodule algebras

Glauber Quadros and Felipe Castro

Abstract

It will be defined a globalization for a partial H-bicomodule algebra, extending the notion given by E. Batista and M. Muniz Alves [1]. It will be also shown that every partial H-bicomodule algebra has a globalization and if H^0 separate points, there is an isomorphism of H^0 bimodule algebras between the standard globalization for partial H-bicomodule algebra and the standard globalization for partial H^0-bimodule algebra.

Bicomodule Algebra

A $\mathbb{K}$-algebra B is a left H-comodule algebra if exists a linear map $\lambda : B \to H \otimes B$ such that:

(i) $\lambda(ab) = \lambda(a)\lambda(b)$;

(ii) $(\varepsilon \otimes I_A)\lambda(a) = 1_{\mathbb{K}} \otimes a$;

(iii) $(I_A \otimes \lambda)\lambda(a) = (\Delta \otimes I_A)\lambda(a)$;

We can define a right H-comodule algebra in a similar way, using a linear map $\rho : B \to B \otimes H$.

We say that B is an H-bicomodule algebra if it is a left and right H-comodule algebra and $(I_H \otimes \rho)\lambda = (\lambda \otimes I_H)\rho$.

We usually denote $\lambda(a) = a^{-1} \otimes a^{-0}$ and $\rho(a) = a^{+0} \otimes a^{+1}$.

Examples

Example 1: Let A a unital algebra. Then A is an H-bicomodule algebra with the trivial coactions given by $\rho(a) = a \otimes 1_H$ and $\lambda(a) = 1_H \otimes a$.

Example 2: H is an H-bicomodule algebra with coactions given by $\rho = \lambda = \Delta_H$.

Partial Bicomodule Algebra

A $\mathbb{K}$-algebra A is a left partial H-comodule algebra if exists a linear map $\lambda : A \to H \otimes A$ such that:

(i) $\lambda(ab) = \lambda(a)\lambda(b)$

(ii) $(\varepsilon \otimes I_A)\lambda(a) = 1_{\mathbb{K}} \otimes a$

(iii) $(I_A \otimes \lambda)\lambda(a) = (\Delta \otimes I_A)\lambda(a)(1_H \otimes \lambda(1_A))$ for all a, b in A.

We can define a right partial H-comodule algebra in a similar way, using a linear map $\rho : A \to A \otimes H$.

We say that A is a partial H-bicomodule algebra if it is a left and right partial H-comodule algebra and $(I_H \otimes \rho)\lambda = (\lambda \otimes I_H)\rho$.

Examples

Example 1: Let G a finite group and A an algebra over a field K. Let G_1 and G_2 two subgroups of G such that their respective orders do not divide the characteristic of K.Then A is a partial KG-bicomodule algebra with coactions

$$\rho : A \to A \otimes KG, \quad a \mapsto a \otimes \sum_{g \in G_1} \frac{1}{|G_1|} g \qquad \text{and } \lambda : A \to KG \otimes A, \quad a \mapsto \sum_{g \in G_2} \frac{1}{|G_2|} g \otimes a \,.$$

Example 2: Let $H_4 = \langle g, x | g^2 = 1, x^2 = 0, xg = -gx\rangle$ the Sweedler algebra over a field K with characteristic different of 2. Then K becomes a partial H_4-bicomodule algebra with coactions defined by

$$\rho : K \to K \otimes H_4, \quad 1_K \mapsto 1_K \otimes \tfrac{1}{2}1 + \tfrac{1}{2}g + \alpha xg \qquad \text{and } \lambda : K \to H_4 \otimes K, \quad 1_K \mapsto \tfrac{1}{2}1 + \tfrac{1}{2}g + \beta x \otimes 1_K$$

where $\alpha, \beta \in K$.

Correspondence Theorems

Theorem 1: If A is a right H-comodule algebra with coaction ρ, then A is a left H^0-module algebra with action $f \triangleright a = a^{+0} f(a^{+1})$. Analogously if A is a left H-comodule algebra with coaction λ, then A is a left H^0-module algebra with action $a \triangleleft f = a^{-0} f(a^{-1})$.

Theorem 2: Let H be a Hopf Algebra such that H^0 separate points, A a K-algebra and $\rho : A \to A \otimes H$ a linear map. If A is a left (partial) H^0-module algebra with the action given by $f \triangleright a = a^{+0} f(a^{+1})$ so A is a right (partial) H-comodule algebra by ρ. A similar result works for right actions and left coactions.

Theorem 3: Let A be an algebra, the linear maps ρ, λ and the induced maps $\triangleright$, $\triangleleft$ like above. If H^0 separate points, then A is a (partial) H-bicomodule algebra with coactions ρ and λ if and only if A is a (partial) H^0-bimodule algebra with the induced actions $\triangleright$ and $\triangleleft$.

Correspondence Between Induced Actions and Induced Coactions

Suppose H is a Hopf algebra such that H^0 separate points. Let B be an H-bicomodule algebra and consider in B the induced structure of H^0-bimodule algebra. If exists a unital subalgebra A of B, then the following statements are equivalent:

- $(a \triangleleft f)(g \triangleright b) = (a \triangleleft f)1_A(g \triangleright b)$ in A, $\forall a, b \in A$ and $f, g \in H^0$
- $(\lambda(a) \otimes 1_H)(1_H \otimes \rho(b)) = (\lambda(a) \otimes 1_H)(1_H \otimes 1_A \otimes 1_H)(1_H \otimes \rho(b))$ in $H \otimes A \otimes H$, $\forall a, b \in A$.

Induced Partial Coaction

Let B be an H-bicomodule algebra and A a unital subalgebra of B such that $(\lambda(a) \otimes 1_H)(1_H \otimes \rho(b)) = (\lambda(a) \otimes 1_H)(1_H \otimes 1_A \otimes 1_H)(1_H \otimes \rho(b))$ in $H \otimes A \otimes H$, $\forall a, b \in A$. So A becomes a partial H-bicomodule algebra with coactions given by

$$\bar{\rho} : A \to A \otimes H, \quad a \mapsto (1_A \otimes 1_H)\rho(a) \qquad \bar{\lambda} : A \to H \otimes A, \quad a \mapsto \lambda(a)(1_H \otimes 1_A)$$

Globalization

Given a unital partial H-bicomodule algebra A with partial coactions $\bar{\rho}, \bar{\lambda}$, a pair (B, θ) is a globalization for A, with coactions ρ, λ, if $\theta : A \to B$ is an algebra monomorphism and

- B is the H-bicomodule algebra (not necessarily unital) generated by $\theta(A)$, i.e., the smallest H-bicomodule algebra containing $\theta(A)$
- $(\lambda(\theta(a)) \otimes 1_H)(1_H \otimes \rho(\theta(b))) = (I_H \otimes \theta \otimes I_H)((\bar{\lambda}(\phi(a)) \otimes 1_H)(1_H \otimes \bar{\rho}(\phi(b))))$ for all $a, b \in A$.

Some Consequences of Globalization

If B is a globalization of A, so we have some properties:

(i) $(\theta \otimes I_H)(\bar{\rho}(a)) = (\theta(1_A) \otimes 1_H)\rho(\theta(a))$

(ii) $(I_H \otimes \theta)(\bar{\lambda}(a)) = \lambda(\theta(a))(1_H \otimes \theta(1_A))$

(iii) $(I_H \otimes \theta \otimes I_H)((\bar{\lambda} \otimes I_H)\bar{\rho}(a)) = (1_H \otimes \theta(1_A) \otimes 1_H)[(\lambda \otimes I_H)\rho(\theta(a))](1_H \otimes \theta(1_A) \otimes 1_H) = (1_H \otimes \theta(1_A) \otimes 1_H)[(I_H \otimes \rho)\lambda(\theta(a))](1_H \otimes \theta(1_A) \otimes 1_H) = (I_H \otimes \theta \otimes I_H)((I_H \otimes \bar{\rho})\bar{\lambda}(a))$

A Foundation Lemma

Let B be an H-bicomodule algebra with coactions ρ and λ, $A \subseteq B$ a subalgebra and $S := H^* \triangleright A \triangleleft H^*$ an H^*-bimodule where the actions are induced by the coactions. Then the algebra generated by S is the smallest H-bicomodule subalgebra of B containing A.

Standard Globalization

Let A be a partial H-bicomodule algebra by $(\bar{\rho}, \bar{\lambda})$ and take $\mathcal{X} := H \otimes A \otimes H$.

Note that $\mathcal{X}$ is an H-bicomodule algebra with coactions

$$\rho : \mathcal{X} \to \mathcal{X} \otimes H, \quad x \mapsto (I_H \otimes I_A \otimes \Delta_H)(x) \qquad \text{and } \lambda : \mathcal{X} \to H \otimes \mathcal{X}, \quad x \mapsto (\Delta_H \otimes I_A \otimes I_H)(x)$$

Moreover, there exists a multiplicative monomorphism from A to $\mathcal{X}$ defined by:

$$\phi : A \to \mathcal{X}, \quad a \mapsto (I_H \otimes \bar{\rho})\bar{\lambda}(a)$$

such that $(\lambda(\phi(a)) \otimes 1_H)(1_H \otimes \rho(\phi(b))) = (I_H \otimes \phi \otimes I_H)((\bar{\lambda}(\phi(a)) \otimes 1_H)(1_H \otimes \bar{\rho}(\phi(b))))$ $\forall a, b \in A$.

Using the above result, we obtain that the algebra B generated by $H^* \triangleright \phi(A) \triangleleft H^*$ is an H-bicomodule algebra and so it is a globalization for the partial H-bicomodule algebra A. We call this globalization the Standard globalization for a partial H-bimodule algebra. And this construction shows the following result:

Theorem: Every partial H-bicomodule algebra has a globalization.

Correspondence Between Globalizations for Partial H-Bicomodule Algebra and Partial H^0-Bimodule Algebra

Note that $\mathcal{X}$ is an H^0-bimodule algebra with the induced actions, i.e.,

$$\begin{aligned} f \triangleright (h \otimes a \otimes k) &:= (h \otimes a \otimes k)^{+0} f((h \otimes a \otimes k)^{+1}) \\ &= (h \otimes a \otimes k_1 f(k_2)) \end{aligned}$$

$$\begin{aligned} (h \otimes a \otimes k) \triangleleft f &:= f((h \otimes a \otimes k)^{-1})(h \otimes a \otimes k)^{-0} \\ &= f(h_1)h_2 \otimes a \otimes k \end{aligned}$$

Now we define the linear map

$$\Psi : H \otimes A \otimes H \to Hom(H^0 \otimes H^0, A), \quad h \otimes a \otimes k \mapsto \Psi(h \otimes a \otimes k)(f \otimes g) := f(h)ag(k) \,.$$

The map Ψ is an algebra homomorphism and an H^0-bimodule map. Moreover, $\Psi\phi = \varphi$, where φ is the standard globalization for the induced partial H^0-bimodule algebra A. If H^0 separate points, Ψ is injective.

So we obtain a new result:

Theorem: If H^0 separate points, then the standard globalization of A as a partial H-bicomodule algebra and the standard globalization of A as partial H^0-bimodule algebra are isomorphic as H^0-bimodule algebras.

References

[1] Marcelo Muniz S. Alves and Eliezer Batista, *Globalization theorems for partial Hopf (co)actions, and some of their applications*, Contemp. Math. **537** (2011), 13–30.

[2] ———, *Enveloping actions for partial Hopf actions*, Comm. Algebra **38** (2010), no. 8, 2872–2902.

[3] ———, *Partial Hopf actions, partial invariants and a Morita context*, Algebra Discrete Math. **3** (2009), 1–19.

[4] S. Caenepeel and K. Janssen, *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*, Comm. Algebra **36** (2008), no. 8, 2923–2946.

[5] Michael Dokuchaev, Miguel Ferrero, and Antonio Paques, *Partial actions and Galois theory*, J. Pure Appl. Algebra **208** (2007), no. 1, 77–87.

UFRGS UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL

INSTITUTO DE MATEMÁTICA UFRGS

CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico